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The electromagnetic field of solenoids with time-dependent currents

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Abstract. Electromagnetic fields of the toroidal solenoid with a time-dependent current are studied. Their properties and the possible practical applications of the results obtained are discussed.

1. Introduction

Treating the Aharonov-Bohm effect, one frequently encounters a situation in which the magnetic flux inside the solenoid S is time dependent. The non-zero electric field arising outside S changes the angular momentum of a charged particle [1]. Usually it is suggested [2] that the magnetic field H is equal to zero outside S. This fact has recently been disputed in [3-5]. It was noted in [3] that if the electric field E changes with time outside S, then necessarily $H \neq 0$ there. This leads to the scattering of charged particles in this magnetic field. It was proved in [4] that the absence of H outside S is consistent with the Maxwell equations if the magnetic flux ϕ inside S does not depend on time or has linear time dependence. Finally, it was stated in [5] that instant switching on of the current in an infinite solenoid leads to a cylindrical wave expanding with the light velocity. Those papers are purely of a qualitative nature; no concrete expressions for fields are given. Naturally, the electromagnetic fields of solenoids were studied earlier without reference to the Aharonov-Bohm effect. Two interesting papers by Miller [6, 7] should be mentioned, in which the correct temporal dependence of the solenoid electromagnetic field was guessed. The present treatment aims to justify and realize the qualitative considerations of [3-7] using the toroidal solenoid as an example. The physical consequences of the time-dependent current flowing through the winding of an infinite cylindrical solenoid were studied in [8, 9]. The plan of our exposition is as follows. In section 2 we study the properties of the toroidal solenoid with a constant current in its winding. It turns out that the magnetic field vanishes inside this solenoid for a very specific current density. In section 3 we obtain vector potentials (VP), field strengths and Poynting vectors for a number of temporal dependences of the solenoid current. In section 4 we discuss the possible applications of the results obtained in the previous two sections.

2. Some facts on the toroidal solenoid with a constant current

Let the solenoid winding be performed on the torus

$$(\rho - d)^2 + z^2 = R^2. \tag{2.1}$$

Its surface may be parametrized as follows:

$$x=(d+R\cos\psi)\cos\varphi \qquad y=(d+R\cos\psi)\sin\varphi$$
$$z=R\sin\psi \qquad (0<\varphi<2\pi, 0<\psi<2\pi).$$

The infinitesimal surface element is equal to $dS=R(d+R\cos\psi)\,d\psi\,d\varphi$. In what follows we shall frequently use the toroidal coordinates

$$x=a\frac{\sinh\mu\cos\varphi}{\cosh\mu-\cos\theta} \qquad y=\frac{a\sinh\mu\sin\varphi}{\cosh\mu-\cos\theta} \qquad z=a\frac{\sin\theta}{\cosh\mu-\cos\theta}$$
$$0<\mu<\infty \qquad -\pi<\theta<\pi \qquad 0<\varphi<2\pi. \tag{2.2}$$

For μ fixed, points $P(x, y, z)$ fill the surface of the torus with parameters $d=a\coth\mu$, $R=a/\sinh\mu$. Let $\mu=\mu_0$ correspond to the torus T. Then for $\mu>\mu_0$ (respectively, $\mu<\mu_0$) the point $P(x, y, z)$ (where x, y, z are given by (2.2)) lies inside (respectively, outside) T. The infinitesimal volume element is expressed in toroidal coordinates as

$$dV=a^3\frac{\sinh\mu\,d\mu\,d\theta\,d\varphi}{(\cosh\mu-\cos\theta)^3}.$$

We shall distinguish two types of the solenoid winding. First, when each particular coil of the winding lies in the plane $\varphi=$ constant. We call this type of winding as a meridional one. Second, when each particular coil lies in the plane $z=$ constant. We refer to this as to latitude winding. Sometimes these two types of winding are called toroidal and poloidal, respectively (see, e.g., Status Report on Controlled Thermonuclear Fusion, Vienna 1990). Consider these two cases separately.

2.1. *Meridional winding*

In the stationary case the magnetic field H is equal to zero outside the solenoid S and $H_\rho=H_z=0$, $H_\varphi=g/\rho$ inside it. The constant g may be expressed through either the total number of coils N and current I ($g=2NI/c$) or the magnetic flux inside S $g=(\phi/2\pi)(d-\sqrt{d^2-R^2})^{-1}$. For the constant I the VP components in the Coulomb gauge were written explicitly in [10]. Later they were used for the electron scattering studies on the toroidal solenoid [11]. We now briefly review their properties. Only two cylindrical components (A_ρ and A_z) are different from zero: $A_\rho=gA_\rho^{(0)}$, $A_z=gA_z^{(0)}$ where

$$A_z^{(0)}=\frac{\sqrt{R}}{2\pi}\int_0^{2\pi}d\varphi\frac{d-\rho\cos\varphi}{[(\rho\cos\varphi-d)^2+z^2]^{3/4}}Q_{1/2}(\cosh\nu)$$
$$A_\rho^{(0)}=\frac{\sqrt{R}}{2\pi}z\int_0^{2\pi}d\varphi\frac{\cos\varphi}{[(\rho\cos\varphi-d)^2+z^2]^{3/4}}Q_{1/2}(\cosh\nu) \tag{2.3}$$
$$\cosh\nu=\frac{r^2+d^2+R^2-2d\rho\cos\varphi}{2R[(\rho\cos\varphi-d)^2+z^2]^{1/2}} \qquad r^2=\rho^2+z^2.$$

A_z and A_ρ are even and odd functions of z, respectively. Further, A_ρ vanishes on the z axis and in the $z=0$ plane. At large distances from the solenoid A_z and A_ρ fall as r^{-3}†:

$$A_z\approx\frac{1}{8}\pi gdR^2\frac{1+3\cos2\theta_s}{r^3} \qquad A_\rho\approx\frac{3}{8}\pi gdR^2\frac{\sin2\theta_s}{r^3}.$$

† r and θ_s are the usual spherical coordinates.

Consider now the A_z behaviour in the $z=0$ plane. At the origin

$$A_z = g\sqrt{\frac{R}{d}}\,Q_{1/2}\left(\frac{d^2+R^2}{2Rd}\right).$$

Inside the solenoid (for $\rho=\sqrt{d^2-R^2}$)

$$A_z = gP_{-1/2}\left(\frac{d}{R}\right)Q_{1/2}\left(\frac{d}{R}\right).$$

For large values of ρ it is negative: $A_z=-\pi gdR^2/4\rho^3$. On the z axis

$$A_z = \frac{g\sqrt{R}d}{(d^2+z^2)^{3/4}}\,Q_{1/2}\left(\frac{d^2+z^2+R^2}{2R\sqrt{d^2+z^2}}\right)$$

which goes to $\pi gdR^2/2|z|^3$ for large values of z. For the thin solenoid ($R \ll d$) the integrals (2.3) can be taken in a closed form. Outside S one obtains

$$\begin{aligned} A_z^{(0)} &= \frac{R^2}{2(d\rho)^{3/2}}\frac{1}{\sinh\mu_1}[\rho Q^1_{1/2}(\cosh\mu_1) - dQ^1_{-1/2}(\cosh\mu_1)] \\ A_\rho^{(0)} &= -\frac{R^2 z}{2(d\rho)^{3/2}}\frac{1}{\sinh\mu_1}Q^1_{1/2}(\cosh\mu_1) \qquad \cosh\mu_1 = \frac{r^2+d^2}{2d\rho}. \end{aligned} \tag{2.4}$$

For the thin solenoid in the $z=0$ plane A_z grows from $\pi gR^2/2d^2$ at the origin up to $gR/2d$ at the inner boundary ($\rho=d-R$) of the solenoid. Inside it A_z vanishes approximately at $\rho=d$ and takes the negative value $-gR/2d$ at the outer ($\rho=d+R$) solenoid boundary. For larger values of ρ A_z remains negative and goes to zero: $A_z=-\pi gdR^2/4\rho^3$. Consider now $A_z(\rho, z)$ as a function of z for ρ fixed. On the z axis $A_z=\pi gR^2d/2(z^2+d^2)^{3/2}$. For fixed $\rho<d$, A_z is positive for all z; for $\rho>d$ it is negative for small z and positive for larger z. Zeros of A_z in the (ρ, z) plane lie on the curve which originates at the point $(d, 0)$ and has asymptotes $z=\pm\rho/\sqrt{2}$. This behaviour of A_z reflects the fact that $\int_{-\infty}^{\infty} A_z(\rho, z)\,\mathrm{d}z$ is equal to the magnetic field flux if the integration path passes through the solenoid hole ($\rho<d$) and zero otherwise ($\rho>d$).

2.2. *The latitudinal winding*

The contribution to the VP of a particular coil lying on the torus surface ($\psi=$ constant or $z=$ constant) equals:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{j}{c}\oint\frac{\boldsymbol{n}_{\varphi'}}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}\varphi'. \tag{2.5}$$

Here j is the current in a particular coil, $\boldsymbol{n}_\varphi$ is a unit vector defining the current direction in this coil ($\boldsymbol{n}_\varphi=\boldsymbol{n}_y\cos\varphi-\boldsymbol{n}_x\sin\varphi$). The single non-vanishing component of the VP is

$$A_\varphi(\rho, z) = \frac{2j}{c\sqrt{\rho s}}\,Q_{1/2}\left(\frac{t^2}{2\rho s}\right). \tag{2.6}$$

Here $s=d+R\cos\psi$, $t^2=\rho^2+(d+R\cos\psi)^2+(z-R\sin\psi)^2$; $Q_\nu(x)$ is the Legendre function of the second kind. Integrating (2.6) with respect to ψ, one obtains the VP of the solenoid with a latitude direction of the current:

$$A_\varphi(\rho, z) = \frac{2j}{c\sqrt{\rho}}\int\frac{\mathrm{d}\psi}{\sqrt{s}}\,Q_{1/2}\left(\frac{t^2}{2\rho s}\right). \tag{2.7}$$

At large distances it falls as r^{-2}

$$A_\varphi \approx \frac{2\pi^2 jd \sin\theta_s}{cr^2}$$

(r and θ_s are the usual spherical coordinates). In deriving (2.7) we implicitly assumed the uniform distribution of coils over the torus surface (or more precisely, the coil density in the $\varphi=$ constant plane is taken to be independent of the angle ψ). The current density corresponding to VP (2.7) is determined by applying to it the Laplacian: $\Delta \boldsymbol{A} = -(4\pi/c)\boldsymbol{j}$. This gives

$$\boldsymbol{j} = \boldsymbol{n}_\varphi j \frac{\delta[\sqrt{(\rho-d)^2+z^2}-R]}{R(d+R\cos\psi)} = \boldsymbol{n}_\varphi \frac{(\cosh\mu_0-\cos\theta)^2}{a^3}\delta(\mu-\mu_0)$$
$$R = a/\sinh\mu_0 \qquad d = a\coth\mu_0. \tag{2.8}$$

It is convenient to rewrite VP (2.7) in the toroidal coordinates

$$A_\varphi = -\frac{8\sqrt{2}}{ca} j \sinh\mu_0(\cosh\mu-\cos\theta)^{1/2}\sum\frac{\cos n\theta}{1+\delta_{n0}}\frac{f_n(\mu,\mu_0)}{n^2-\frac{1}{4}}Q_{n-1/2}(0). \tag{2.9}$$

Here function $f_n(\mu,\mu_0)$ equals $P^{(1)}_{n-1/2}(0)Q^{(1)}_{n-1/2}$ inside the solenoid ($\mu>\mu_0$) and $Q^{(1)}_{n-1/2}(0)P^{(1)}_{n-1/2}$ outside it ($\mu<\mu_0$); $P^\lambda_\nu(x)$ and $Q^\lambda_\nu(x)$ are associated Legendre functions of the first and second kind, respectively; further, $P^\lambda_\nu(0) \equiv P^\lambda_\nu(\cosh\mu_0)$ and $Q^\lambda_\nu(0) \equiv Q^\lambda_\nu(\cosh\mu_0)$. In this equation and in the following ones we do not indicate the argument of the Legendre functions if it is $\cosh\mu$. In addition, the summation, if not otherwise stated, is extended from $n=0$ to $n=\infty$. It follows from (2.7) or (2.9) that the magnetic field H does not vanish inside the solenoid.

2.3. *On the disappearance of magnetic field inside the solenoid*

On the other hand, it is stated frequently (see, e.g., [7, 12]) that for the current flowing in the latitude direction, magnetic field H vanishes inside the solenoid. Now we specify the conditions under which this takes place. For this we present the equation $H = \operatorname{rot}\boldsymbol{A} = 0$ in toroidal coordinates. Bearing in mind that A_φ is the single non-vanishing component of the VP, we obtain

$$H_\mu = \frac{(\cosh\mu-\cos\theta)^2}{a\sinh\mu}\frac{\partial}{\partial\theta}\left(\frac{\sinh\mu\, A_\varphi}{\cosh\mu-\cos\theta}\right)$$
$$H_\theta = -\frac{(\cosh\mu-\cos\theta)^2}{a\sinh\mu}\frac{\partial}{\partial\mu}\left(\frac{\sinh\mu\, A_\varphi}{\cosh\mu-\cos\theta}\right) \qquad H_\varphi \equiv 0.$$

From this it follows at once that $H=0$ inside the solenoid ($\mu>\mu_0$) if

$$A_\varphi = \frac{\cosh\mu-\cos\theta}{\sinh\mu}C_0 \qquad C_0 = \text{constant}. \tag{2.10}$$

Clearly, A_φ should satisfy the Poisson equation. We present this equation in the form

$$\Delta \boldsymbol{A} = -\frac{4\pi}{c}\delta(\mu-\mu_0)j(\theta)\boldsymbol{n}_\varphi. \tag{2.11}$$

Here j_θ is a single-valued function to be determined later. The finite solution of this equation is

$$A_\varphi = (\cosh\mu - \cos\theta)^{1/2} \sum D_n \frac{1}{1+\delta_{n0}} P^{(1)}_{n-1/2} \cos n\theta \tag{2.12}$$

outside the solenoid ($\mu < \mu_0$) and

$$A_\varphi = (\cosh\mu - \cos\theta)^{1/2} \sum F_n \frac{1}{1+\delta_{n0}} Q^{(1)}_{n-1/2} \cos n\theta \tag{2.13}$$

inside it. The physical VP should be continuous everywhere and particularly at $\mu = \mu_0$. This gives

$$F_n = D_n P^{(1)}_{n-1/2}(0) / Q^{(1)}_{n-1/2}(0).$$

Further, inside the solenoid ($\mu > \mu_0$), (2.13) should coincide with (2.10). This determines F_n:

$$F_n = C_0 \frac{\sqrt{2}}{\pi} \frac{1}{n^2 - \frac{1}{4}}.$$

Thus, we have

$$A_\varphi = C_0 \frac{\cosh\mu - \cos\theta}{\sinh\mu}$$

$$= C_0 \frac{\sqrt{2}}{\pi} (\cosh\mu - \cos\theta)^{1/2} \sum \frac{\cos n\theta}{1+\delta_{n0}} \frac{1}{n^2 - \frac{1}{4}} Q^{(1)}_{n-1/2} \tag{2.14}$$

inside the solenoid and

$$A_\varphi = C_0 \frac{\sqrt{2}}{\pi} (\cosh\mu - \cos\theta)^{1/2} \sum \frac{\cos n\theta}{1+\delta_{n0}} \frac{1}{n^2 - \frac{1}{4}} \frac{Q^{(1)}_{n-1/2}(0)}{P^{(1)}_{n-1/2}(0)} P^{(1)}_{n-1/2} \tag{2.15}$$

outside it. According to (2.11), the discontinuity of $\partial A_\varphi / \partial\mu$ at $\mu = \mu_0$ determines $j(\theta)$:

$$j(\theta) = -\frac{cC_0}{2\sqrt{2}\pi^2 a^2} \frac{(\cosh\mu_0 - \cos\theta)^{5/2}}{\sinh\mu_0} \sum \frac{\cos n\theta}{1+\delta_{n0}} \frac{1}{P^{(1)}_{n-1/2}(0)}. \tag{2.16}$$

To this current density then corresponds the distribution of the number of coils, which is proportional to

$$(\cosh\mu_0 - \cos\theta)^{1/2} \sum \frac{\cos n\theta}{1+\delta_{n0}} [P^{(1)}_{n-1/2}(0)]^{-1}. \tag{2.17}$$

At large distances A_φ falls as r^{-2}:

$$A_\varphi \approx \frac{2a^2}{\pi r^2} C_0 \sin\theta_s \sum \frac{1}{1+\delta_{n0}} \frac{Q^{(1)}_{n-1/2}(0)}{P^{(1)}_{n-1/2}(0)}.$$

We conclude that the disappearance of H inside the solenoid when the current is directed latitudinally takes place for the very specific current density (2.16) or for the distribution of the number of coils (2.17).

3. Toroidal solenoid with a time-dependent current†

Let the current in the solenoid winding change with time. Contrary to the cylindrical solenoid case [8, 9], the electromagnetic fields of the toroidal solenoid do not have postaction properties. This means that all transient effects (which are due to the sudden switching on of the current in the solenoid) come to an end at a given point P as soon as the action from the most remote solenoid point arrives at P. Consider a few concrete forms of temporal dependences.

3.1. $g = j_0\theta(t)$

For

$$t > t_1 \equiv \frac{[(\rho+d)^2+z^2]^{1/2}+R}{c}$$

the VP at the point $P(\rho, z, \varphi)$ is

$$A_z = j_0 A_z^{(0)} \qquad A_\rho = j_0 A_\rho^{(0)} \tag{3.1}$$

where $A_z^{(0)}$ and $A_\rho^{(0)}$ are given by (2.3). Their properties were discussed in the previous section. Now we have the following physical picture. For the fixed time t the field strengths and Poynting vector differ from zero inside the shell

$$-d\sin\theta_s + [(ct-R)^2 - d^2\cos^2\theta_s]^{1/2} \leqslant r \leqslant d\sin\theta_s + [(ct+R)^2 - d^2\cos^2\theta_s]^{1/2}.$$

This shell has a width of $\sim 2R$ along the z axis and a width of $2(d+R)$ along the x or y axes. It expands at the velocity of light. In front of this shell ($r \geqslant [(ct+R)^2 - d^2\cos^2\theta_s]^{1/2} + d\sin\theta_s$) the vector potential is equal to zero. Behind it ($0 \leqslant r \leqslant [(ct-R)^2 - d^2\cos^2\theta_s]^{1/2} - d\sin\theta_s$ it is given by (3.1).

3.2. $g(t) = 0$ for $t < 0$ and $g = j_1 t$ for $t > 0$

Then, for $t > t_1$ we have $A_\rho = j_1 t A_\rho^{(0)}$, $A_z = j_1 t A_z^{(0)}$ where $A_\rho^{(0)}$ and $A_z^{(0)}$ have been defined above. Outside S only the electric field strengths differ from zero: $E_\rho = -(1/c) j_1 A_\rho^{(0)}$, $E_z = -(1/c) j_1 A_z^{(0)}$. Thus, there is no electromagnetic energy flow into the surrounding space for the toroidal solenoid with a linearly growing current. The Poynting flux is concentrated inside S near its surface and directed inside the solenoid. For simplicity, we prove this for the thin solenoid only. The VP inside S, expressed in toroidal coordinates, is given by [10]

$$A_\rho = j_1 t \exp(-\mu)\sin\theta \qquad A_z = -j_1 t \exp(-\mu)\cos\theta. \tag{3.2}$$

It follows from this that only the θ component of A differs from zero inside solenoids: $A_\theta = -j_1 t \exp(-\mu)$ (for the thin solenoid $\exp(-\mu) = [(\rho-d)^2+z^2]^{1/2}/2d$). Non-vanishing field strengths equal $E_\theta = (1/c) j_1 \exp(-\mu)$, $H_\varphi = j_1 t/\rho$. The Poynting vector is directed towards the solenoid equatorial line ($\rho = d$, $z = 0$): $S = S_\mu e_\mu$, $S_\mu = j_1 t \exp(-\mu)/4\pi\rho c^2$. Let a closed resistive loop C pass through the solenoids hole. The constant electric field outside S induces the current $j = \sigma E_\theta$ (σ is the wire conductivity) in C. As a result, Joule heat is produced in C. We know that the energy flow for $t > t_1$ is zero outside S. In view of this fact it is unclear how the energy is transferred from

† The results of this section are referred to solenoids with a meridional winding.

the solenoid to the resistive loop C. The following intricate answer was found in [13]. The current flowing in C induces magnetic field H. Its lines of force are directed along the circumferences which surround C. As a result, the Poynting flux $(1/4\pi c)E_\theta H$ arises, which is perpendicular to C and which 'flows' into C. In other words, it is just the current induced in C that gives rise to a non-zero energy flow.

The analysis of the following gedanken experiment is very instructive. Let a wire loop C pass through the hole of the toroidal solenoid in which the linearly growing current flows. This loop may be part of a resistance bridge with an external battery chosen in such a way as to compensate the current induced in C by the solenoid constant electric field. It is known [14, 15] that the resistance of the current loop threaded by the magnetic flux is a periodic function of this flux value. In view of this, the balance of the bridge will be periodically disturbed. This may be registered with some device. Now, a question arises as to how the energy is transferred from the solenoid to this device? According to the reasoning of [13], this proceeds along the following lines. The change of the solenoid magnetic flux changes the resistance of C. As a result, a uncompensated current appears in C, which produces the magnetic field H concentric to C. The Poynting vector constructed from the electric field E_θ and induced magnetic field H is perpendicular to C and may be viewed as a candidate for the energy transfer.

3.3. $g = j_2 t^2 \theta(t)$

For $t > t_1$

$$A_\rho = j_2 t^2 A_\rho^{(0)} + \frac{1}{c^2} j_2 A_\rho^{(2)} \qquad A_z = j_2 t^2 A_z^{(0)} + \frac{1}{c^2} j_2 A_z^{(2)}.$$

Here $A_\rho^{(0)}$ and $A_z^{(0)}$ have been defined earlier, while $A_\rho^{(2)}$ and $A_z^{(2)}$ are given by

$$A_z^{(2)} = \int \mathrm{d}\varphi (d - \rho \cos\varphi) F(\rho, z, \varphi) \qquad A_\rho^{(2)} = z \int \cos\varphi \, F(\rho, z, \varphi)$$

$$F(\rho, z, \varphi) = \frac{R^{3/2}}{4\pi} \frac{1}{[(\rho \cos\varphi - d)^2 + z^2]^{1/4}} [Q_{3/2}(\cosh \nu) - Q_{-1/2}(\cosh \nu)].$$

At large distances $A_z^{(2)}$ and $A_\rho^{(2)}$ fall as r^{-1}:

$$A_z^{(2)} \approx -\frac{\pi R^2 d}{8r}(3 + \cos 2\theta_s) \qquad A_\rho^{(2)} \approx -\frac{\pi R^2 d}{8r} \sin 2\theta_s.$$

Further, $A_\rho^{(2)}$ equals zero on the z axis and in the $z = 0$ plane. On the z axis $A_z^{(2)}$ is of the form

$$A_z^{(2)}(\rho = 0, z) = \frac{1}{2} \frac{R^{3/2} d}{(d^2 + z^2)^{1/4}} [Q_{3/2}(\cosh \mu_0) - Q_{-1/2}(\cosh \mu_0)]$$

$$\cosh \mu_0 = \frac{z^2 + d^2 + R^2}{2R\sqrt{d^2 + z^2}}.$$

For the thin solenoid the VP components are simplified:

$$A_z^{(2)} = -\frac{1}{2} \frac{R^2}{(d\rho)^{1/2}} [d Q_{-1/2}(\cosh \mu_1) - \rho Q_{1/2}(\cosh \mu_1)]$$

$$A_\rho^{(2)} = -\frac{1}{2} \frac{R^2}{(d\rho)^{1/2}} z Q_{1/2}(\cosh \mu_1) \qquad \cosh \mu_1 = \frac{r^2 + d^2}{2d\rho}.$$

The following field strengths are different from zero:

$$E_z = -\frac{R^2 t j_2}{c(d\rho)^{3/2}} \frac{1}{\sinh \mu_1} [\rho Q^1_{1/2}(\cosh \mu_1) - d Q^1_{-1/2}(\cosh \mu_1)]$$

$$E_\rho = \frac{R^2 z t j_2}{c(d\rho)^{3/2}} \frac{1}{\sinh \mu_1} Q^1_{1/2}(\cosh \mu_1)$$

$$H_\varphi = -\frac{R^2}{c^2} \frac{j_2}{(d\rho)^{1/2}} Q_{1/2}(\cosh \mu_1).$$

We observe that outside the solenoid there is a magnetic field which does not depend upon the time. For $r \to \infty$ one gets

$$E_\rho \approx -\frac{3\pi d R^2 j_2 t}{4cr^3} \sin 2\theta_s \qquad E_z \approx -\frac{\pi d R^2 j_2 t}{4cr^3}(1 + 3\cos 2\theta_s)$$

$$H_\varphi = -\frac{\pi d R^2 j_2}{2c^2 r^2} \sin \theta_s .$$

The energy flow through the spherical surface of a sufficiently large radius is directed off the solenoid

$$S_r = \frac{1}{4\pi c} E_\theta H_\varphi = \frac{\pi d^2 R^4 j_2^2 t \sin^2 \theta_s}{16 c^4 r^5}.$$

3.4. $g(t) = j_3 t^3 \theta(t)$

Then, for $t > t_1$ one has

$$A_\rho = j_3 t^3 A_\rho^{(0)} + \frac{3}{c^2} j_3 t A_\rho^{(2)} \qquad A_z = j_3 t^3 A_z^{(0)} + \frac{3}{c^2} j_3 t A_z^{(2)} + \frac{\pi R^2 d j_3}{c^3}.$$

The non-vanishing field strengths are

$$E_\rho = -\frac{3}{c} j_3 t^2 A_\rho^{(0)} - \frac{3}{c^3} j_3 A_\rho^{(2)} \qquad H_\varphi = -\frac{3R^2 j_3 t}{c^2 (d\rho)^{1/2}} Q_{1/2}(\cosh \mu_1)$$

$$E_z = -\frac{3 d_3 t^2}{c} A_z^{(0)} - \frac{3 j_3}{c^3} A_z^{(2)}.$$

For large distances

$$E_\theta \approx -\frac{3\pi d R^2 \sin \theta_s j_3}{4c^3 r}\left(1 + \frac{c^2 t^2}{r^2}\right) \qquad H_\varphi \approx -\frac{3 j_3 t \pi d R^2 \sin \theta_s}{2c^2 r^2}.$$

The radial component of the Poynting vector is directed off the solenoid:

$$S_r = \frac{9\pi j_3 t d^2 R^4 \sin^2 \theta_s}{32 c^6 r^3}\left(1 + \frac{c^2 t^2}{r^2}\right).$$

3.5. The solenoid current is a periodical function of time: $g = g_0 \cos \omega t$

For small solenoid dimensions ($R/r \ll 1$, $KR \ll 1$, $d/r \ll 1$, $Kd \ll 1$) the VP equals

$$A_r = \frac{\Lambda \cos \theta_s}{2r^3} (\cos \psi + Kr \sin \psi)$$

$$A_\theta = \frac{\Lambda \sin \theta_s}{4r^3} [(1 - K^2 r^2) \cos \psi + Kr \sin \psi]$$

$$\Lambda = \pi g_0 d R^2 \qquad \psi = Kr - \omega t \qquad K = \omega / c.$$

The non-vanishing electromagnetic strengths are

$$E_r = \frac{K\Lambda \cos \theta_s}{2r^3} (Kr \cos \psi - \sin \psi) \qquad H_\varphi = \frac{K^2 \Lambda \sin \theta_s}{2r^2} (Kr \sin \psi + \cos \psi)$$

$$E_\theta = \frac{K\Lambda \sin \theta_s}{4r^3} [(K^2 r^2 - 1) \sin \psi + Kr \cos \psi].$$

At large distances from the solenoid ($Kr \gg 1$, i.e. in the wave zone) one obtains

$$E_r = \frac{K^2 \Lambda}{2r^2} \cos \theta_s \cos \psi \qquad E_\theta = H_\varphi = \frac{K^3 \Lambda \sin \theta_s \sin \psi}{4r}.$$

These limiting values of field strengths should coincide with those radiated by the toroidal magnetic moment [16]. This indeed takes place. The radial component of the Poynting vector is directed off the solenoid:

$$S_r = \frac{1}{4\pi c} E_\theta H_\varphi = \frac{1}{4\pi c} (E_\theta)^2.$$

The integral energy flow through the sphere of sufficiently large radius is

$$r^2 \int S_r \, d\Omega = \frac{1}{24c} (\Lambda K^3 \sin \psi)^2.$$

We conclude: time dependence of the solenoid current generally leads to a non-vanishing magnetic field outside the solenoid. The flow of the electromagnetic energy is directed off the solenoid.

4. Conclusion

There can be a twofold intepretation of the results obtained. First, they can be viewed as an exercise in mathematical physics. In fact, the remarkable properties of the toroidal solenoid are practically unknown and the matter presented here and in [10] fill this lacuna. From the non-vanishing of the magnetic field outside the solenoid it follows that the transition process to the stationary regime in the AB effect theory should be reconsidered. Do the results presented have more practical use? There are branches of applied physics in which enormous magnetic fields are stored at first and spent later. One of these branches is the electromagnetic launch technology (symposia on this subject are held regularly and their proceedings are published in IEEE Transactions on Magnetics. The last one devoted to this subject is volume 25, no 1 (1989)). The results of section 3 show that the current flowing in the winding of the toroidal solenoid (in which the magnetic field is accumulated) should vary linearly with time in order to escape the irreversible electromagnetic energy losses into the surrounding space. The second possible application is the construction of a pulse transformer for electromagnetic launching. It was shown in [12] that a toroidal transformer with a vanishing magnetic field inside it ('external field toroid' according to terminology of [12]) can transfer 99.9% of the stored energy into the secondary winding, thus greatly reducing compressive forces (which tend to destroy a transformer's windings). The results of subsection 2.3 show that a very specific type of winding should be used in order that H should vanish inside the toroidal solenoid. The third branch in which the obtained results can be applied is the toroidal plasma and especially the electromagnetic field in a Tokamak. The use of explicit expressions for the VP obtained in [10] and in the present work can simplify the complicated system of equations used for the description of the toroidal plasma (see, e.g., [17]).

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